A NOTE ON CONSTRUCTION OF SOME SYMMETRICAL PBIB DESIGNS WITH DUAL PROPERTY

By

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1. Introdcction

An m-associate class symmetrical PBIB design ($v, k, \lambda_i, n_i, = l, 2 \ldots m$) is said to possess the dual property if its dual obtained by interchanging the blocks with treatments also possesses the same association scheme as the original design. Bose [1] has defined and discussed Symmetrical group divisible designs (SGDD) possessing the dual property. The above definition as a generalization of his definition for a SGDD with dual property. In this note we have presented the construction of three-associate class symmetrical PBIB designs with dual property having the Rectangular Association Scheme defined by Vartak (4) and have derived the conditions under which these degenerate into lower associate class designs.

A PBIB design is said to posses the rectangular association scheme if it contains $v=mn$ symbols (treatments) each replicated r times in b blocks of size k units, which can be arranged in an mxn array such that any two symbols are first associations if they occur together in the same row of the array, second association if they occur in the same column and third association otherwise. The designs based on the above association scheme are called Rectangular Designs. When $v=b$, the design is said to be symmetrical Rectangular (SG) Design.

For definitions of the other designs we refer to Raghava Rao [2].

2. Results

Several methods of constructing rectangular designs are available in literature. The general method based on the sum of kronecker products of designs put forward by Surendran [3] has its wide applicability in construction of these designs. In this paper we have presented a method of construction of symmetrical rectangular (SR) designs with dual property by developing an initial block formed by certain symbols occuring in the same row and same column of the array with a particular symbol. Incidentally the design so formed also turns out to be a special case of the designs obtainable from the above method.

3. Method of Construction

Let there be an $m \times n$ array with elements

$$\begin{matrix} 1 & m+1 & 2m+1 \ldots & (n-1)\,m+1 \\ 2 & m+2 & 2m+2 & (m-1)\,m+2 \\ m & 2m & 3m & nm \end{matrix} \qquad (\ldots 1)$$

and an associated symmetrical PBIB design $(4t+3,\ 2t+1t,)$ such that $m=4t+3$, where t is an integer and $4t+3$ *a* prime power.

It is well known that the above design can be constructed by developing mod $(4t+3)$ the initial block $(x^{\circ}, x^2, \ldots x4t)$, x being a primitive element of G.F. $(4t+3)$ (See Raghava Rao [2].

Consider an initial block containing $2t+1$ symbols selected from the first column of the array corresponding to a block of the associated SBIB design and all symbols which occur in the same row with a particular symbol (treatment) also belonging to the first column, but different from the symbol included in the initial block. Without loss of generality we can form an initial block with contents.

$$(x^0+1,\ x^2+1 \ldots x^{4t}+1;\ m+1, 2m+1 \ldots \overline{n-1},\ m+1). \qquad (\ldots 2)$$

the first $2t+1$ elements of which correspond to block of the SBIB design given above and occur in the first column of the array while the rest of $n-1$ elements occur in the first row. Clearly neither of the portions of the initial block contains the first symbol though its contents occur with it either in the row or the column.

Let a set of m blocks be obtained by adding $1, 2, \ldots m-1$ successively to the initial block at (2) with the contents of first portion reduced mod (m), then we have

Theorem

The set of m blocks formed as above together with $n-1$ sets each of m blocks obtained by adding mod (mn) the numbers $m, 2m, \ldots$ $(n-1)\,m$ to the above set generate a symmetrical rectangular (SP) design with dual property having parameters

$$\left.\begin{array}{l} v=(4t+3)\ n=b,\ r=2t\ n=k, \\ \lambda 1=n-2,\ \lambda 2=t,\ \lambda 3=1 \\ n_1=n-1, n_2=m-1, n_3=(m-1)(m-1) \end{array}\right) \quad \ldots(3)$$

Proof:

It is easy to see that the incidence matrix of the constructed design assumes the form

$$N=In\times N_1+(Jn-In)\times Im \quad \ldots(4)$$

Where N_1 is the incidence matrix of the SBIB design whose first block which also serves as an initial block in this case is $(x^0+1, x^2+1 \ldots x^{4t}+1)$. It being a square matrix of order t, Jt a square matrix of order t having unity everywhere and x denotes the Kronecker product of matrices.

Hence following Surendran [3], the design at (4) is a PBIB design having at the most 1+1+1,1=3 associate classes, Also $N'N$

$$=In\ x(N'_1N_1+(n-1)\ \ Im)+(Jn-In)x(N'_1+N_1+(n-2)Im) \quad \ldots(5)$$

To prove the dual property of the design we have to show that $N_1'+N_1$ is symmetric. It is noteworthy that N_1 whose blocks are obtained by adding 0,1,2,...mod $(4t+3)$ to the initial block given as above has all zeros in its diagonal since the ith row (block) taken in that order does not contain the i th treatment $(i=1,2,\ldots 4t+3)$. Also $N_1\neq N_1'$

Now consider the initial block

$$(x^0+1,\ x^2+1,\ \ldots\ldots x^{4t}+1,\ 1) \quad \ldots(6)$$

of size $2t+2$. Noting that $x^{2t}+1=-1$, it is easy to verify that among the differences arising out of the above set, all the non-null elements of G.F. $(4t+3)$ occur exactly $t+1$ times. Consequently the above initial block generates a SBIBD $(4t+3, 2t+2, t+1)$. If N_2 is the incidence matrix of this design, we have atonce

$$N_1+I_m=N_2$$

Hence $$(N_1'+I_m)\ (N_1+I_m)=N_2'\ N_2 \quad \ldots(8)$$

giving $$N_1'\ N_1+(N_1'+N_1)+I_m=N_2'\ N_2 \quad \ldots(9)$$

since I_m is a unit matrix.

Now

$$\left.\begin{array}{l}(i)\ N_1'\ N_1=(2t+1-t)\ Im+t\ Jm=(t+1)\ Im+t\ Jm\\(ii)\ N_2'\ N_2=(2t+2-t)\ Im+(t+1)\ Jm=(t+1)\ Im+Jm)\end{array}\right\}\quad ...(10)$$

Using (10) in (9) we readily obtain

$$N_1'+N_1=Jm-Im \quad ...(11)$$

As a check, multiplying (11) by Jm we get

$$2(2t+1)\ Jm=(m-1)\ Jm,$$

giving

$$m=4t+3$$

From (5), (10) and (11) we get

$$N'\ N=Im\times[(t+n)\ Im+t\ Jm)]+(Jm-Im)\times[(n-3)\ Im+Jm)]$$

$$=N\ N' \quad ...(12)$$

as all its components are symmetric matrices ...(13)

Hence the design possesses the dual property.

Again, in order that N may be the incidence matrix of a SR design ($v=mn$, k, λ_1, λ_2, λ_3) it will be evident from a little consideration of the association scheme that $N\ N$ should be of the form

$$N'\ N=Im\times[(k-\lambda_2)\ Im+\lambda_2\ Jm)]+(Jm-Im)\times(\lambda_1-\lambda_3)\ Im+\lambda_3\ Jm)] \quad ...(14)$$

Comparing the coefficients of In and $(Jn-In)$ in (12) and (14) we get

$$\left.\begin{array}{l}(k-\lambda_2)\ Im+\lambda_2\ Jm=(t+n)\ Im+t\ Jm\\(\lambda_1-\lambda_3)\ Im+\lambda_3\ Jm=(n-3)Im+Jm\end{array}\right\}\quad ...(15)$$

Equating the coefficients of Im & Jm

$$\left.\begin{array}{r}K-\lambda_2=t+1\\\lambda_2=t\\\lambda_1-\lambda_3=n-3\\\lambda_3=1\end{array}\right\}\quad ...(16)$$

from which all the relations at (3) follow. This completes the proof.

The above design degenerates into a symmetrical Group Divisible design (SGDD) with dual property when $t=n-2\neq 1$.

In particular, when $n-2=t=1$ *i.e.* $n=3$ and $t=1$, we get the SBIBD (21, 5, 1).

4. Illustrative Example

As an example consider a 7×2 array, the associated SBIBD (7, 3, 1) and the initial block (2, 3, 5, 8).

Obtained in the manner discussed above, 3 being the primitive element of G.F.(7). From this block we obtain a SGDD having dual property with parameters.

$$v=14=b,$$
$$r=4=k,$$
$$\lambda_1=0,$$
$$\lambda_2=0,$$
$$n_1=1,$$
$$n_2=12.$$

The blocks with their contents are

(2, 3, 5, 8), (9, 10, 12, 1)
(3, 4, 6, 9), (10, 11, 13, 2)
(4, 5, 7, 10), (11, 12, 14, 3)
(5, 6, 1, 11), (12, 13, 8, 4)
(6, 7, 2, 12), (13, 14, 9, 5)
(7, 1, 3, 13), (14, 8, 10, 6)
(1, 2, 4, 14), (8, 9, 11, 7)

The group of first associates being given by the set containing ith and $(1+7)$th treatments $(i=1, 2, \ldots 7)$.

Corollary

In case the initial block(2) is formed by taking all the symbols occuring in the first column & first row with the first symbol of the array, the mn blocks obtained in the manner followed in theorem 1, generate a SR design with dual property. The parameters of the design are

$$\left.\begin{array}{l} v=mn=b,\ r=m+n-2=k,\ \lambda_1=n-2,\ \lambda_2=m-2 \\ \lambda_3=2,\ n_1=n-1,\ n_2=m-1,\ n_3=(m-1)(n-1) \end{array}\right\} \qquad \ldots(18)$$

Proof

The proof follows from theorem 1 noting that the design generated by the column elements of the array contained in the initial block is the trivial SBIB design $(m, m-1, m-2)$ whose incidence matrix is

$$M=M'=Jm-Im \qquad \ldots(19)$$

The above design becomes (*i*) a SGDD with dual property when $m=4$, $n\neq 4$, (*ii*) a design having L_2 (*iii*) association scheme when $m=n\neq 4$ and (*iii*) a SBIB design when $m=n=4$.

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